Journal of Sound and Vibration (2001) **244**(2), 359–366
doi:10.1006/jsvi.2000.3465, available online at http://www.idealibrary.com on IDEAL®

LETTERS TO THE EDITOR

AN APPROACH TO DIRECT AND INVERSE TIME-DOMAIN SCATTERING OF ACOUSTIC WAVES FROM RIGID POROUS MATERIALS BY A FRACTIONAL CALCULUS BASED METHOD

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(*Received* 13 *March* 2000, *and in final form* 4 *September* 2000)

1. INTRODUCTION

In this paper direct and inverse time-domain scattering of ultrasonic pulses from a rigid, homogeneous and isotropic porous medium are investigated. The Green function of the wave propagation of a transient field in one-dimensional porous media is established. The solutions of direct and inverse problems are given in the time domain by using the concept of fractional derivatives. The viscous and thermal losses of the medium are described by the Johnson *et al.* and Allard models [1, 2] modified to be usable in the time domain. Experimental and numerical results are given as a validation of our model.

2. MODEL

The determination of the properties of a medium from waves that have been reflected by or transmitted through the medium is a classical inverse scattering problem. Such problems are often approached by taking a physical model of the scattering process generating a synthetic response for some assumed values of the parameters, adjusting these parameters until reasonable agreement is obtained between the synthetic response and the observed data. Most publications concerned with such acoustical investigations are devoted to frequency-domain methods. However, because of the transient nature of signals and to avoid the computation of numerous Fourier transforms, it is more appropriate to compare the synthetic signal and the data in the time domain. There are several other relevant reasons for dealing with time-domain techniques: (i) they allow the rapid acquisition of data over a large band width; (ii) they allow the separation of differents events by time gating in the time domain; (iii) a time-domain model is often the most natural description of the way in which the actual experiment is performed.

In the acoustics of porous materials, one distinguishes two situations according to whether the frame is moving or not. In the first case, the dynamics of the waves due to the

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coupling between the solid skeleton and the fluid is well described by the Biot theory [3, 4]. In air-saturated porous media the structure is generally motionless and the waves propagate only in the fluid. This case is described by the model of an equivalent fluid which is a particular case of the Biot model, in which the interactions between the fluid and the structure are taken into account in two frequency-dependent response factors: the dynamic tortuosity of the medium $\alpha(\omega)$ given by Johnson *et al.* [1] and the dynamic compressibility of the air included in the porous material $\beta(\omega)$ given by Allard [2]. In the frequency domain, these factors multiply the density of the fluid and its compressibility respectively and represent the deviation from the behaviour of the fluid in free space as the frequency increases. In the time domain, they act as operators and in the high frequency approximation their expressions are given by Fellah and Depollier [5] as

$$\tilde{\alpha}(t) = \alpha_\infty \left(\delta(t) + \frac{2}{\Lambda} \left(\frac{\eta}{\rho_f} \right)^{1/2} t^{-1/2} \right), \tag{1}$$

$$\tilde{\beta}(t) = \left(\delta(t) + \frac{2(\gamma - 1)}{\Lambda'} \left(\frac{\eta}{\mathrm{Pr}\rho_f} \right)^{1/2} t^{-1/2} \right). \tag{2}$$

In these equations, $\delta(t)$ is the Dirac function, Pr is the Prandtl number, η and ρ_f are respectively the fluid viscosity and the fluid density and γ is the adiabatic constant. The relevant physical parameters of the model are the tortuosity of the medium α_∞ and the viscous and thermal characteristic lengths Λ and Λ' introduced by Johnson *et al.* [1] and Allard [2]. In this model $t^{-1/2}$ is interpreted as a semi-derivative operator following the definition of the fractional derivative of order ν given by Samko *et al.* [6],

$$D^{\nu}[x(t)] = \frac{1}{\Gamma(-\nu)} \int_0^t (t-u)^{-\nu-1} x(u)\, \mathrm{d}u, \tag{3}$$

where $\Gamma(x)$ is the gamma function.

In this framework, the basic equations of the model can be written as

$$\rho_f \tilde{\alpha}(\mathrm{t}) * \frac{\partial v_i}{\partial t} = -\nabla_{iP} \quad \text{and} \quad \frac{\tilde{\beta}(t)}{K_a} * \frac{\partial p}{\partial t} = -\nabla \cdot v, \tag{4}$$

where * denotes the time convolution operation, p is the acoustic pressure, v is the particle velocity and K_a is the bulk modulus of the air. The first equation is the Euler equation, and the second one is a constitutive equation obtained from the equation of mass conservation associated with the behaviour (or adiabatic) equation.

For a wave propagating along the x-axis, these equations become

$$\rho_f \alpha_\infty \frac{\partial v}{\partial t} + 2 \frac{\rho_f \alpha_\infty}{\Lambda} \left(\frac{\eta}{\pi \rho_f} \right)^{1/2} \int_{-\infty}^{t} \frac{\partial v / \partial t'}{\sqrt{t - t'}} \mathrm{d}t' = -\frac{\partial p}{\partial x}, \tag{5}$$

$$\frac{1}{K_a} \frac{\partial p}{\partial t} + 2 \frac{\gamma - 1}{K_a \Lambda'} \left(\frac{\eta}{\pi \mathrm{Pr} \rho_f} \right)^{1/2} \int_{-\infty}^{t} \frac{\partial p / \partial t'}{\sqrt{t - t'}} \mathrm{d}t' = -\frac{\partial v}{\partial x}. \tag{6}$$

In these equations the convolutions express the dispersive nature of the porous material. They take into account the memory effects due to the fact that the response of the medium to the wave excitation is not instantaneous but needs more time to become effective. The retarding force is no longer proportional to the time derivative of the acoustic velocity but is found to be proportional to the fractional derivative of order $\frac{1}{2}$ of this quantity. This occurs because the volume of fluid participating in the motion is not the same during the whole length of the signal as it is in the case of a fully developed steady flow. The phenomenon may be understood by considering such a volume of fluid in a pore to be in harmonic motion. At high frequencies, only a thin layer of fluid is excited: the average shear stress is high. At a lower frequency, the same amplitude of fluid motion allows a thicker layer of fluid to participate in the motion and consequently, the shear stress is less. The penetration distance of the viscous forces and therefore the excitation of the fluid depends on frequency. In the time domain, such a dependence is associated with a fractional derivative.

3. DIRECT PROBLEM

The direct scattering problem is that of determining the scattered field as well as the internal field, that arises when a known incident field impinges on the porous material with known physical properties. To compute the solution of the direct problem one needs to know the Green function of the modified wave equation in the porous medium. In that case, the internal field is given by the time convolution of the Green function with the incident wave and the reflected and transmitted fields are deduced from the internal field and the boundary conditions.

The generalized lossy wave equation in the time domain is derived from the basic equations (5) and (6) by elementary calculation in the form

$$\frac{\partial^2 p}{\partial x^2} - A\frac{\partial^2 p}{\partial t^2} - B\int_{-\infty}^{t} \frac{\partial^2 p/\partial t'^2}{\sqrt{t-t'}}\,\mathrm{d}t' - C\frac{\partial p}{\partial t} = 0, \tag{7}$$

where the coefficients A, B and C are constants respectively given by

$$A = \frac{\rho_f \alpha_\infty}{K_a}, \qquad B = \frac{2\alpha_\infty}{K_a}\sqrt{\frac{\rho_f \eta}{\pi}}\left(\frac{1}{\Lambda} + \frac{\gamma - 1}{\sqrt{\mathrm{Pr}}\Lambda'}\right), \qquad C = \frac{4\alpha_\infty(\gamma-1)\eta}{K_a \Lambda\Lambda'\sqrt{\mathrm{Pr}}}. \tag{8}$$

The first one is related to the velocity $c = 1/\sqrt{\rho_f \alpha_\infty / K_a}$ of the wave in the air included in the porous material. α_∞ appears as the refractive index of the medium which changes the wave speed from $c_0 = \sqrt{K_a/\rho_f}$ in free space to $c = c_0/\sqrt{\alpha_\infty}$ in the porous medium. The other coefficients are essentially dependent on the characteristic lengths Λ and Λ' and express the viscous and thermal interactions between the fluid and the structure. The constant B governs the spreading of the signal while C is responsible for the attenuation of the wave. Obviously, a knowledge of these three coefficients allows the determination of the parameters α_∞, Λ and Λ'. One way to solve equation (7) with suitable initial and boundary conditions is by using the Laplace transform. The approach is quite simple although the inverse Laplace transform requires tedious calculus [6]. A suitable setting for the introduction of the time-domain solution of the modified wave propagation equation (7) is provided by the following model. Consider a homogeneous porous medium which fills the half space $x \geqslant 0$ and an incident signal $g^i(t)$ which impinges normally on the surface $x = 0$ from the left at time $t = 0$. For porous media having a high porosity like plastic foams, the reflected signal can be neglected. These materials have such a small amount of rigid frame that the incident wave does not feel its effects. In that case, the direct problem lies in finding the solution of equation (7) with the following boundary and initial conditions

$$p(0,t) = g^i(t) \quad \text{and} \quad \lim_{\substack{t\to 0\\ t>0}} p(x,t) = \lim_{\substack{t\to 0\\ t>0}} \frac{\partial p}{\partial t}(x,t) = 0. \tag{9}$$

The initial conditions mean in physics that the medium is idle for $t = 0$. The solution of the propagation equation (7) is given by the convolution of the Green function $G(x, t)$ with the input signal $g^i(t)$,

$$p(x,t) = \int_0^t G(x, t-t')g^i(t')\,\mathrm{d}t'. \tag{10}$$

Within the porous medium, the Green function of the direct problem is given by the expression

$$G(x,t) = \left\{\begin{matrix} 0 & \text{if } 0 \leqslant t \leqslant x/c \\ \dfrac{x}{c}\dfrac{b'}{4\sqrt{\pi}}\dfrac{1}{(t-x/c)^{3/2}}\exp\left(-\dfrac{b'^2x^2}{16c^2(t-x/c)}\right) + \Delta\int_0^{t-x/c} h(t,\xi)\,\mathrm{d}\xi & \text{if } t \geqslant x/c \end{matrix}\right\}, \tag{11}$$

where $h(\tau, \xi)$ is of the form

$$h(\tau,\xi) = -\frac{1}{4\pi^{3/2}}\frac{1}{\sqrt{(\tau-\xi)^2 - x^2/c^2}}\frac{1}{\xi^{3/2}}\int_{-1}^{1}\exp\left(-\frac{\chi(\mu,\tau,\xi)}{2}\right)(\chi(\mu,\tau,\xi)-1)\frac{\mu\,\mathrm{d}\mu}{\sqrt{1-\mu^2}}, \tag{12}$$

with the notations $\chi(\mu,\tau,\xi) = (\Delta\mu\sqrt{(\tau-\xi)^2 - x^2/c^2} + b'(\tau-\xi))^2/8\xi$, $b' = Bc^2\sqrt{\pi}$, $c' = C.c^2$ and $\Delta = b'^2 - 4c'$. It is easy to show that this solution is continuous on the surface $x = 0$ of the porous material:

$$\lim_{x\to 0} p(x,t) = p(0,t) = g^i(t). \tag{13}$$

As an application of the model, some numerical simulations are compared to experimental results. The simulated signals are computed from equation (10) in which $g^i(t)$ is the signal given out by the transducer. The experimental data are deduced from the transmitted field scattered by a slab of plastic foam of finite depth $0 \leqslant x \leqslant L$. In dealing with a slab of high porosity foam, as already mentioned above, the signals reflected by the front wall ($x = 0$) and by the back wall ($x = L$) of the slab can be neglected. Thus, near the back wall, the signal propagating in the foam is nearly identical to the transmitted one $p(L-\varepsilon, t) = g^t(L+\varepsilon, t)$. For foams having low porosity this approximation breaks down and in that case, reflected signals must be taken into account [7, 8]. Experiments are performed in air with two broadband Panametrics V389 piezoelectric transducers having a 200 kHz central frequency in air and a bandwidth at 6 dB extending from 60 to 420 kHz. Pulses of 900 V are provided by a 5058PR Panametrics pulser/receiver (see Figure 1). Received signals are amplified up to 90 dB and filtered above 1 MHz to avoid

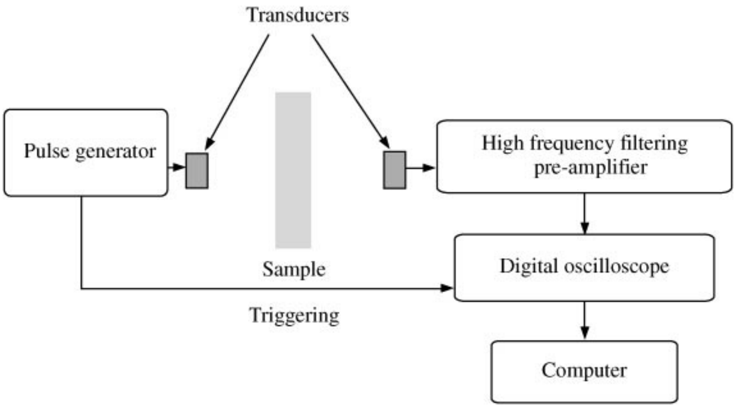

Figure 1. Experimental set-up of the ultrasonic measurements.

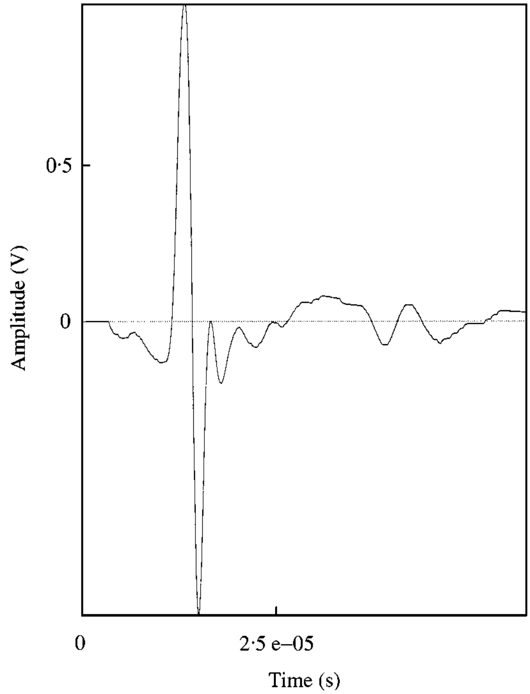

Figure 2. Incident signal given out by the transducer.

high-frequency noise. Figure 2 shows the incident signal given out by the transducer. In Figure 3, experimental and simulated results are presented for two plastic foams F_1 and F_2 having different flow resistivities. The parameters of the foam F_1 are: thickness 5 cm, $\alpha_\infty = 1{\cdot}055$, $\Lambda = 234\,\mu$m, $\Lambda' = 702\,\mu$m, flow resistivity $\sigma = 9000\,\mathrm{Nm^{-4}\,s}$ and porosity $\phi = 0{\cdot}97$; those of the foam F_2 are: thickness 1·1 cm, $\alpha_\infty = 1{\cdot}26$, $\Lambda = 60\,\mu$m, $\Lambda' = 180\,\mu$m, $\sigma = 38\,000\,\mathrm{Nm^{-4}\,s}$ and $\phi = 0{\cdot}98$. The good agreement for foams with low or high flow resistivity, especially for the maximum value of their amplitudes, may be regarded as being in support of the quite realistic assumption about the replacement of the transmitted signal

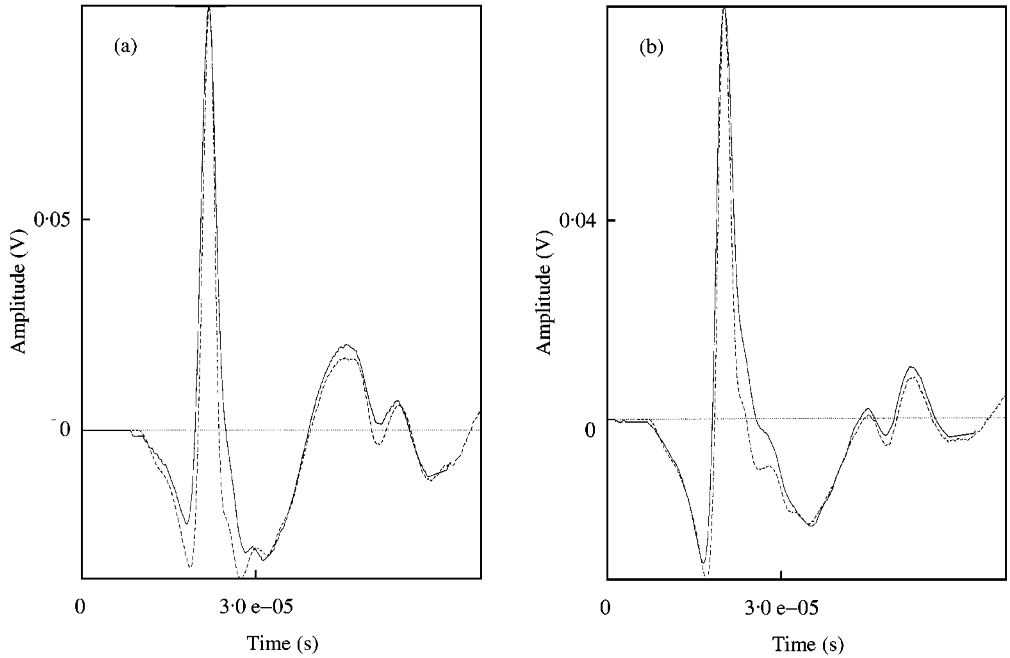

Figure 3. (a) Experimental (solid line) and simulated signals (dashed line) for the foam F1; (b) experimental (solid line) and simulated signals (dashed line) for the foam F2.

by the internal one. The slight difference observed between the two curves is probably due to experimental measurements rather than to the lack of reflection on the walls of the slab.

4. INVERSE PROBLEM

The interior of the slab of porous material is characterized by three parameters, α_∞, Λ and Λ', the values of which are crucial for the behaviour of the sound waves. So, it is of some importance to work out new experimental methods and efficient tools for their estimation. Therefore, a basic inverse problem associated with the slab may be stated as follows: from the measurements of the transmitted signals outside the slab determine the parameters of the medium. As shown in section 2, the solution of the direct problem can be considered as a three-parameter family of functions (the coefficients A, B and C can be expressed in α_∞, Λ and Λ'). The problem of finding the values of the parameters of the slab can be formulated as a fitting problem: find the values of the parameters α_∞, Λ and Λ' such that the transmitted signal describes the scattering problem in the best possible way (e.g., in the least-squares sense).

The inverse problem is to find values of coefficients α_∞, Λ and Λ' which minimize the function

$$U(A, B, C) = \int_0^T (g^t(t) - p(L, t))^2 \, \mathrm{d}t \tag{14}$$

where $g^t(t)$ and $p(L, t)$ are respectively the experimental transmitted signal, and the solution of the wave equation (10) near the back wall of the slab, and T is the length of the signal.

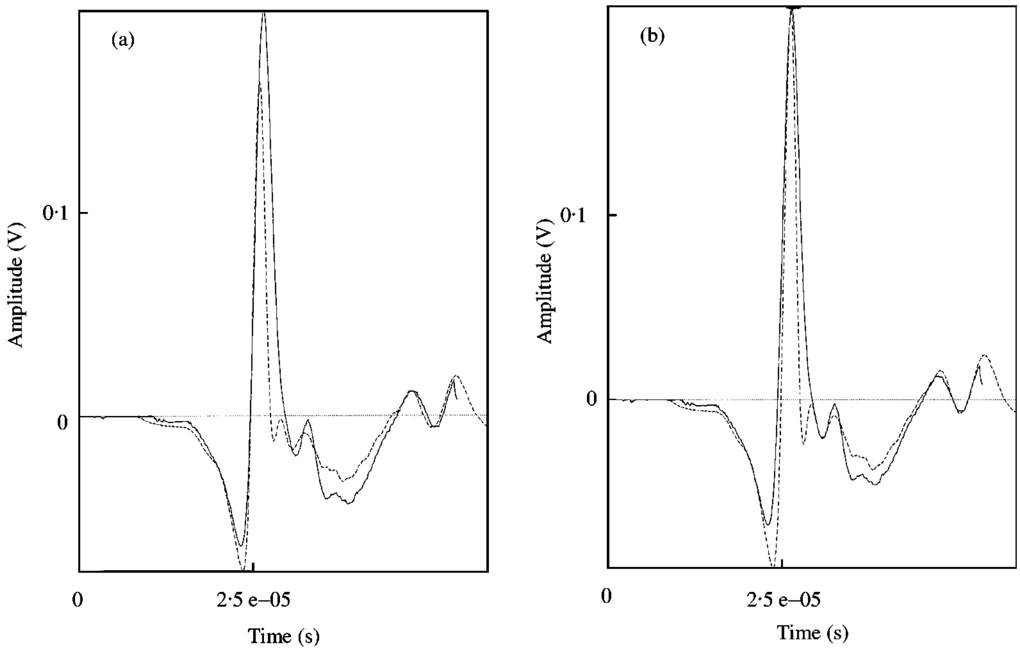

Figure 4. (a) Experimental (solid line) and simulated signals (dashed line) with the parameters given by the Leclaire method [9] for the foam F3; (b) experimental (solid line) and simulated signals (dashed line) with the parameters given by the authors' method.

However, because of the non-linearity of the equations and of the numerous local minima, the solution of the inverse problem by the conventional least-squares methods is tedious. In our case, one can seek the numerical solution which minimizes the $U(A, B, C)$ defined by

$$U(A, B, C) = \sum_{i=1}^{N} (g_i^t - p(x, t_i))^2, \tag{15}$$

where $g_i^t = g^t(t_i)_{i=1,2,\ldots,N}$ represents the discrete set of values of the transmitted signal and $p(x, t_i)_{i=1,2,\ldots,N}$ represents the discrete set of values of the solution $p(x, t)$ as a function of A, B and C.

The values of the parameters of the material given by the inverse problem are $\alpha_\infty = 1{\cdot}062$, $\Lambda = 319\ \mu\text{m}$ and $\Lambda' = 957\ \mu\text{m}$. Moreover, for the materials under consideration, the merely slight change of agreement between measurement and simulation under variations of the coefficient C shows that thermal effects are irrelevant for the estimation of the parameters, the best fit being obtained for $\Lambda' \approx 3\Lambda$.

Figure 4 shows the experimental signal transmitted through the plastic foam F_3. The thickness of the slab is equal to 5 cm, the flow resistivity is $\sigma = 2850$ and porosity is $\phi = 0{\cdot}97$. The other parameters that characterize the material are estimated by the classical ultrasonic method [9] ($\alpha_\infty = 1{\cdot}055$, $\Lambda = 300\ \mu\text{m}$ and $\Lambda' = 90\ \mu\text{m}$ (see Figure 4(a)) and by optimization from the inverse problem (see Figure 4(b)). This comparison shows that this time-domain method leads to better results than the previous one and is more efficient in that the criterion for fitting it does not require external intervention.

5. CONCLUSION

In this note, the time domain Green function for the wave equation in porous media is established. The direct problem is solved by using the concept of fractional derivatives and an experimental validation of the model is presented. The physical parameters of the medium are evaluated from the solution of the scattering inverse problem given by a least-squares method. Finaly, a comparison between experimental results and numerical simulation obtained from the optimized parameters shows the efficiency of the method.

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